

To receive full credit, you must show all work.

Question 1 An important function is the function $f(t) = \begin{cases} e^{-1/t^2} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$

a) Show that $f(t) \in C^\infty(\mathbb{R})$. **Hint:** Show that $f^{(n)}(t) = \begin{cases} P(1/t)e^{-1/t^2} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0 \end{cases}$

where $P(z)$ is a polynomial. Then show that $f^{(n)}(t)$ is continuous by taking $\lim_{t \rightarrow 0^+} f^{(n)}(t)$.

Cultural Note: $f(t)$ is an example of a C^∞ function that has a Maclaurin series that converges for all t , but that series only converges to $f(t)$ when $t = 0$. For those who know complex analysis: $f(t)$ is a C^∞ function that is not analytic.

b) Consider the curve $\alpha(t) = (-f(1-t), f(t-1))$, $0 \leq t \leq 2$ in \mathbb{R}^2 . Show that α is C^∞ . What does the curve α look like? Is α regular? Explain. The point of this problem is that the image of a “smooth C^∞ ” function may not look smooth at all. In order to do differential geometry on curves we need that our curves are “locally diffeomorphic” to intervals. This condition implies that the tangent lines to α exist and are well defined. Regularity of α is the condition that guarantees this.

Question 2 Consider the sphere of radius r , $S^2(r) = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2\}$ in \mathbb{R}^3 . The extrinsic distance between points on S^2 is the distance between the points in \mathbb{R}^3 . For example the extrinsic distance between the north and south poles is $2r$. The intrinsic distance d_{in} between points is measured along geodesic arcs (great circles) on the sphere.

a) Find the intrinsic distance between the poles. Now consider the intrinsic metric ball of radius ε centered at the north pole N , $B_{\text{in}}(N, \varepsilon) = \{p \in S^2(r) \mid d_{\text{in}}(N, p) \leq \varepsilon\}$. Let $C_{\text{in}}(\varepsilon)$ and $A_{\text{in}}(\varepsilon)$ denote the circumference and area of $B_{\text{in}}(N, \varepsilon)$, respectively. Find explicit formulas for $C_{\text{in}}(\varepsilon)$ and $A_{\text{in}}(\varepsilon)$.

b) How do they compare to the standard circumference $C_{\text{std}}(\varepsilon)$ and area $A_{\text{std}}(\varepsilon)$ of a circle of radius ε in \mathbb{R}^2 ? Compute the two limits: $\lim_{\varepsilon \rightarrow 0} \frac{C_{\text{in}}(\varepsilon)}{C_{\text{std}}(\varepsilon)}$ and $\lim_{\varepsilon \rightarrow 0} \frac{A_{\text{in}}(\varepsilon)}{A_{\text{std}}(\varepsilon)}$.

c) Compute the two limits:

$$\lim_{\varepsilon \rightarrow 0} \frac{3}{\pi \varepsilon^3} (C_{\text{std}}(\varepsilon) - C_{\text{in}}(\varepsilon))$$

$$\lim_{\varepsilon \rightarrow 0} \frac{12}{\pi \varepsilon^4} (A_{\text{std}}(\varepsilon) - A_{\text{in}}(\varepsilon))$$

Question 3 Recall that the standard Euclidean norm is given by $\|\mathbf{v}\|_{\text{std}}^2 = \mathbf{v} \cdot \mathbf{v}$. One way to change the geometry of a set is to change the norm. Define a new norm on the open ball of radius 2 $= \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\} \subset \mathbb{R}^2$ by

$$\|\mathbf{v}_p\|_{\text{hyp}}^2 = \frac{r^2(\mathbf{v}_p \cdot \mathbf{v}_p)}{[1 - \frac{1}{4}(x^2 + y^2)]^2}, \quad r > 0, \quad p = (x, y)$$

Note that this norm depends not only on the vector, but the point at which the vector is based. Now we get a new notion of length of curve, namely,

$$L_{\text{hyp}}(\alpha) = \int_0^t \|\alpha'(s)\|_{\text{hyp}} ds$$

a) Let $\alpha(s) = (s \cos \theta, s \sin \theta)$, $0 \leq s < 2$, be a ray from the origin. Compute the length of α for $0 \leq s \leq t < 2$. That is, compute the arclength function along α . What happens to the length of α as $t \rightarrow 2$.

b) Compute the circumference $B_{\text{in}}(0, \varepsilon)$ and the limits in problem 2bc that involve the circumference for this new geometric object.

Cultural Note: It can be shown that the limits in 2c and 3b yield a quantity called the curvature of the surface at a point (we may actually be able to show this later). If you did the calculations correctly, you would find that the curvature of a sphere is positive while the curvature you get in problem 3 is negative. In fact, problem 3 is the Poincare disk model of the hyperbolic plane which you may have heard about. It is a geometric object that is topologically equivalent to \mathbb{R}^2 , but has constant negative curvature. Notice also that for all ε , the circumferences satisfy $C_{\text{in}}^{\text{sphere}}(\varepsilon) < C_{\text{std}}(\varepsilon) < C_{\text{in}}^{\text{hyp}}(\varepsilon)$ and the same is true for the areas. Thus, intuitively, the sphere is more cramped than Euclidean space, and Euclidean space in turn is more cramped than hyperbolic space. The limits, for example in 2b, show that asymptotically as $\varepsilon \rightarrow 0$ all Riemannian manifolds look locally like Euclidean space. This is not true, however, for Riemannian orbifolds, the primary objects of some of my research.